One Parameter Semigroups in Two Complex Variables

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October 8, 2022

Midwestern Workshop on Asymptotic Analysis; Purdue University, Fort Wayne

Preliminary Definitions

1. A linear fractional map (aka Mobius transformation) is a complex map given by

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   where $ad-bc \neq 0$.
2. The unit ball in $\mathbb{C}^N$ is given by

$$\mathbb{B}_N = \{z \in \mathbb{C}^N \mid |z| < 1\}.$$

Preliminary Definitions

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A **one parameter semigroup** for a monoid $(S, *)$ is a map $\phi : [0, \infty) \to S$, such that

1. $\phi(0) = I$.
2. $\phi(s + t) = \phi(s) * \phi(t)$.

One Way to Classify Self Maps of the Disk.

For an analytic self-map ϕ of the disk, either ϕ has a fixed point in the disk or else it doesn't. If ϕ has no fixed point in the disk, then there is a unique fixed point α on the boundary with $d(\alpha) \leq 1$. Let $\phi(\alpha) = \alpha$. We classify as follows:

1. Elliptic: $\alpha \in \mathbb{D}$.
2. Hyperbolic: $|\alpha| = 1$ and $d(\alpha) < 1$.
3. Parabolic: $|\alpha| = 1$ and $d(\alpha) = 1$.

where $d(\alpha)$ is the boundary dilation coefficient.

The Fixed-Point Behavior is Important

The one-parameter semigroup $\{\phi_t\}$ depends on its fixed point behavior.

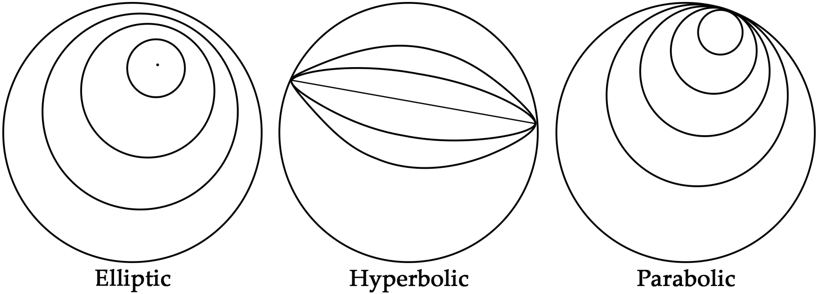

Figure 1: Automorphisms of the Disk

Another Way to Classify Self Maps of the Disk: Part I

(Cowen 1981) Under very general conditions, for an analytic map $\phi : \mathbb{D} \to \mathbb{D}$, there is a domain Ω, either the plane or half-plane, a mapping σ of $\mathbb{D}$ into Ω and a 'model' linear fractional automorphism Φ of Ω such that

$$\sigma \circ \phi = \Phi \circ \sigma$$

We have the following commutative diagram

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\phi} & \mathbb{D} \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ \Omega & \xrightarrow{\Phi} & \Omega \end{array}$$

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$$\sigma \circ \phi = \Phi \circ \sigma \to \sigma \circ \phi_n = \Phi_n \circ \sigma.$$

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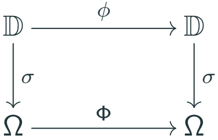

Another Way to Classify Self Maps of the Disk: Part II

After appropriate conjugation, one has the following classification:

1. (Plane/Dilation): $\Omega = \mathbb{C}$, $\sigma(\alpha) = 0$ and $\Phi(z) = sz$ with $0 < |s| < 1$.
2. (Plane/Translation): $\Omega = \mathbb{C}$, $\sigma(\alpha) = \infty$ and $\Phi(z) = z + 1$.
3. (Halfplane/Dilation): $\Omega = \{z \mid \Re z > 0\}$, $\sigma(\alpha) = 0$ and $\Phi(z) = sz$ with $0 < s < 1$.
4. (Halfplane/Translation): $\Omega = \{z \mid \Im z > 0\}$, $\sigma(\alpha) = \infty$ and $\Phi(z) = z \pm 1$.

Example in the Disk.

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Let $\phi(z) = \frac{1}{2}z + \frac{1}{2}$ be a self-map of the disk.

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$$\sigma \circ \phi(z) = 1 - \left(\frac{1}{2}z + \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2}z = \frac{1}{2}(1 - z) = \Phi \circ \sigma(z).$$

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To find $\{\phi_t\}$, we put $\phi_t = \sigma^{-1} \circ \Phi_t \circ \sigma$ where $\Phi_t(z) = \frac{1}{2^t}z$.

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$$\phi_t(z) = \frac{1}{2^t}z + 1 - \frac{1}{2^t}.$$

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Pivot to $\mathbb{B}_N$.

Let ϕ be an analytic self map of $\mathbb{B}_N$.

1. If ϕ is a fixed point free self-map of $\mathbb{B}_N$, then there exists a unique point α on the boundary such that the iterates of ϕ converge uniformly to α on compact subsets of $\mathbb{B}_N$. We call this the Denjoy-Wolff point. (MacCluer 1982)

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2. For the continuous semigroup $\{\phi_t\}$, either all iterates have a common fixed point in $\mathbb{B}_N$ or all iterates (for $t > 0$) have no fixed points in $\mathbb{B}_N$ and share the same Denjoy-Wolff point on the boundary (Abate, 1989).

Definition
We say ϕ is a linear fractional map in $\mathbb{C}^N$ if

$$\phi(z) = \frac{Az + B}{\langle z, C \rangle + D}$$

where A is an $N \times N$ matrix, B and C are column vectors in $\mathbb{C}^N$, $D \in \mathbb{C}$, and $\langle \cdot, \cdot \rangle$ is the standard inner product.

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For an analytic self-map ϕ of $\mathbb{B}_N$ with Denjoy-Wolff point at α, we classify as follows:

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Infinitesimal generators associated with linear fractional self maps of $\mathbb{B}_N$ have been characterized (Bracci, Contreras, Díaz-Madrigal, 2007).

Another Way to Classify Self Maps of the Unit Ball

Our model theory is not adequate for $\mathbb{B}_N$. However, one can show that the model theory extends in $\mathbb{B}_2$ for certain maps (Cowen, et al 2006). The domains are the whole space $\mathbb{C}^N$, half space $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \mid \Re(z_1) > 0\}$, and Siegel half space $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \mid \Re(z_1) > |z_2|^2\}$.

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Our hyperbolic maps admit two.

Our parabolic maps admit four.

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Our hyperbolic maps admit two.

Our parabolic maps admit four.

We are *presuming* that our analytic map ϕ has one of the above models.

Linear Fractional Example

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Let $\phi(z) = \left(\frac{z_1+3}{4}, \frac{z_2}{2}\right)$. What is the Denjoy-Wolff point?

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Let $\phi(z) = \left(\frac{z_1+3}{4}, \frac{z_2}{2}\right)$. What is the Denjoy-Wolff point? To conform to our model we put $\sigma(z) = e_1 - z$ and $\Phi(z) = (\frac{1}{4}z_1, \frac{1}{2}z_2)$ which gives us

$$\sigma \circ \phi(z) = \left(1 - \frac{z_1 + 3}{4}, -\frac{z_2}{2}\right) = \left(\frac{1}{4}(1 - z_1), -\frac{1}{2}z_2\right) = \Phi \circ \sigma(z).$$

The Associatied Matrix and Jordan Canonical Form

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$$m_\phi = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix}.$$

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Recall that Jordan Canonical Form says we can 'factor'

$$m_\phi = S \Lambda S^{-1} \to m_{\phi_n} = (m_\phi)^n = \left(S \Lambda S^{-1}\right)^n = S \Lambda^n S^{-1}.$$

where the columns of S are (generalized) eigenvectors of m_ϕ and Λ is in Jordan form.

Showing $\phi_{t \circ s} = \phi_{t+s}$

Diagonizable Case:

$$m_{\phi_t} = m_\phi^t = S\Lambda^t S^{-1} = S \begin{pmatrix} \lambda_1^t & 0 & 0 \\ 0 & \lambda_2^t & 0 \\ 0 & 0 & \lambda_3^t \end{pmatrix} S^{-1}.$$

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We then have for $s,\ t \geq 0$,

$$m_{\phi_t} m_{\phi_s} = S\Lambda^t S^{-1} S\Lambda^s S^{-1} = S\Lambda^{t+s} S^{-1} = m_{\phi_{t+s}}$$

The Case of a $3x3$ Jordan Block

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Exercise: $\Lambda^t \Lambda^s = \Lambda^{t+s}$.

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$$\phi(z) = \left(\frac{z_1 + 2z_2 + 1}{-z_1 + 2z_2 + 3}, \frac{-2z_1 + 2z_2 + 2}{-z_1 + 2z_2 + 3} \right)$$

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$$= \frac{\begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}}{(-1, 2)^T (z_1, z_2) + 3} = \frac{Az + B}{\langle z, C \rangle + D}.$$

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$$\begin{pmatrix} A & B \\ C^* & D \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 2 & 2 \\ -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{8} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -1 & 2 & 1 \\ -4 & 0 & 4 \end{pmatrix}$$

Example Continued

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$$m_{\phi_t} = \begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{8} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{t}{2} & \frac{t(t-1)}{8} \\ 0 & 1 & \frac{t}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -1 & 2 & 1 \\ -4 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2-t^2}{2} & t & \frac{t^2}{2} \\ -t & 1 & t \\ -\frac{t^2}{2} & t & \frac{t^2+2}{2} \end{pmatrix}$$

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$$\phi_t(z_1, z_2) = \left(\frac{(2-t^2)z_1 + 2tz_2 + t^2}{-t^2 z_1 + 2tz_2 + t^2 + 2}, \frac{-2tz_1 + 2z_2 + 2t}{-t^2 z_1 + 2tz_2 + t^2 + 2} \right).$$

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It is a straightforward calculation to see that $\phi_0 = I$ and $\phi_1 = \phi$.

Nonrational Example

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$$\phi_1(z) = \frac{15z_1 + z_2 + 1 + 4\sqrt{2z_2(z_1+1)} + 4\sqrt{2(1-z_1^2)} + 2\sqrt{z_2(1-z_1)}}{-z_1 + z_2 + 17 + 4\sqrt{2z_2(z_1+1)} + 4\sqrt{2(1-z_1^2)} + 2\sqrt{z_2(1-z_1)}}$$

$$\phi_2(z) = \frac{16z_2 - z_1 + 1 + 8\sqrt{z_2(1-z_1)}}{-z_1 + z_2 + 17 + 4\sqrt{2z_2(z_1+1)} + 4\sqrt{2(1-z_1^2)} + 2\sqrt{z_2(1-z_1).}}$$

Nonrational Example II

Define the following:

$$
\begin{aligned}
A :=& 1024 z_1 + 64 t^2 z_2 + t^2 (t+7)^2 (1 - z_1) + 256 t \sqrt{2 z_2 (z_1 + 1)} \\
&+ 32 t (t+7) \sqrt{2(1 - z_1^2)} + 16 t^2 (t+7) \sqrt{z_2 (1 - z_1)} \\
B :=& 1024 + 64 t^2 z_2 + t^2 (t+7)^2 (1 - z_1) + 256 t \sqrt{2 z_2 (z_1 + 1)} \\
&+ 32 t (t+7) \sqrt{2(1 - z_1^2)} + 16 t^2 (t+7) \sqrt{z_2 (1 - z_1)} \\
C :=& 64 t^2 (1 - z_1) + 1024 z_2 + 512 t \sqrt{z_2 (1 - z_1)} \\
D :=& 1024 + 64 t^2 z_2 + t^2 (t+7)^2 (1 - z_1) + 256 t \sqrt{2 z_2 (z_1 + 1)} \\
&+ 32 t (t+7) \sqrt{2(1 - z_1^2)} + 16 t^2 (t+7) \sqrt{z_2 (1 - z_1)}.
\end{aligned}
$$

Constructing the Semigroup

A calculation shows $\phi_t(z) = (\phi_{1_t}(z), \phi_{2_t}(z))$ where $\phi_{1_t}(z)$ and $\phi_{2_t}(z)$ are given by

$$\phi_{1_t}(z) = \frac{A}{B}$$

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$$\phi_{1_t}(z) = \frac{A}{B}$$

and

$$\phi_{2_t}(z) = \frac{C}{D}$$

A calculation shows that this is a one-parameter semigroup for $\phi : \mathbb{B}_2 \to \mathbb{B}_2$.

Thank You!

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Questions?

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